An Effective Dynamic Programming Algorithm for the Minimum-Cost Maximal Knapsack Packing

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Abstract

Given a set of n items with profits and weights and a knapsack capacity C, we study the problem of finding a maximal knapsack packing that minimizes the profit of selected items. We propose for the first time an effective dynamic programming (DP) algorithm which has $O(nC)$ time complexity and $O(n+C)$ space complexity. We demonstrate the equivalence between this problem and the problem of finding a minimal knapsack cover that maximizes the profit of selected items. In an extensive computational study on a large and diverse set of benchmark instances, we demonstrate that the new DP algorithm outperforms the use of a state-of-the-art commercial mixed integer programming (MIP) solver applied to the two best-performing MIP models from the literature.

1. Introduction

One of the most famous and the most studied problems in combinatorial optimization is the classical *Knapsack Problem* (KP) which is defined as follows: Given an knapsack capacity $C > 0$ and a set $I = \{1, \ldots, n\}$ of items, with profits $p_i \geq 0$ and weights $w_i \geq 0$, KP asks for a maximum profit subset of items whose total weight does not exceed the capacity. The problem can be formulated using the following Integer Linear Program (ILP):

$$\max \left\{ \sum_{i \in I} p_i x_i : \sum_{i \in I} w_i x_i \leq C, x_i \in \{0, 1\}, i \in I \right\} \tag{1}$$

where each variable x_i takes value 1 if and only if item i is inserted in the knapsack. Without loss of generality we can assume that all input parameters have integer values. In the following, we will refer to (I, p, w, C) as a *knapsack instance*. KP is NP-hard, but it is well-known that fairly large instances can be solved to

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optimality within short computing times (see, e.g., [14, 17, 16] for comprehensive surveys on applications and variants of KP).

Optimization problems related to KP search for solutions satisfying one of the following two properties:

Definition 1 (Maximal Knapsack Packing)**.** *Given a knapsack instance (I, p, w, C), a packing $\mathcal{P}$ is a subset of items which does not exceed the capacity C, i.e., $\sum_{i \in \mathcal{P}} w_i \leq C$. A packing $\mathcal{P}$ is called* maximal*, iff $\mathcal{P} \cup \{i\}$ is not a packing for any $i \notin \mathcal{P}$.*

Definition 2 (Minimal Knapsack Cover)**.** *Given a knapsack instance (I, p, w, C), a cover $\mathcal{C}$ is a subset of items which exceeds the capacity C, i.e., $\sum_{c \in \mathcal{C}} w_c \geq C$. A cover $\mathcal{C}$ is called* minimal*, iff $\mathcal{C} \setminus \{i\}$ is not a cover for any $i \in \mathcal{C}$.*

In this article we study weighted versions of the *knapsack covering* and *knapsack packing* problems which are defined below.

Definition 3 (Minimum-Cost Maximal Knapsack Packing (MCMKP))**.** *Given a knapsack instance (I, p, w, C), the goal of the MCMKP is to find a maximal knapsack packing $S^* \subset I$ that minimizes the profit of selected items, i.e.:*

$$S^* = \arg\min_{S \subset I} \{ \sum_{i \in S} p_i \mid \sum_{i \in S} w_i \leq C \text{ and } \sum_{i \in S} w_i + w_j > C, \ \forall j \notin S \}.$$

Definition 4 (Maximum-Profit Minimal Knapsack Cover (MPMKC))**.** *Given a knapsack instance (I, p, w, C), the goal of the MPMKC is to find a minimal knapsack cover $S^* \subseteq I$ that maximizes the profit, i.e.:*

$$S^* = \arg\max_{S \subset I} \{ \sum_{i \in S} p_i \mid \sum_{i \in S} w_i \geq C \text{ and } \sum_{i \in S \setminus \{j\}} w_i < C, \ \forall j \in S \}.$$

In both, MCMKP and MPMKC, we assume that the problem instance (I, p, w, C) is non-trivial, i.e., the corresponding optimal solution S^* is a proper subset of I.

In the classical KP, one tries to pack the knapsack while *maximizing* the (non-negative) profit of selected items, hence, every optimal KP solution trivially satisfies the maximal knapsack packing property. On the contrary, the objective function of the MCMKP goes in the opposite direction – one searches for a subset of items to pack in the knapsack, while *minimizing* the profit of selected items. Hence, solving the standard KP in which the profit-maximization objective is replaced by profit-minimization results in a trivial (empty) solution. This is why the maximal packing property has to be explicitly imposed, when searching for an optimal MCMKP solution.

The MCMKP models applications related to scheduling jobs on a single machine with common arrival times and a common deadline. The problem consists of selecting a subset of jobs to schedule (before the given deadline) and accordingly the order in which the jobs are scheduled becomes irrelevant. The set I refers to all possible jobs, job durations are given as w_i and p_i is the cost for executing job i, $i \in I$. The common deadline C corresponds to the capacity of the knapsack. The goal is to minimize the cost for executing scheduled jobs while at the same time ensuring that the maximal subset of jobs is scheduled (i.e., a maximal packing is preserved). Consequently, the scheduling task boils down to solving MCMKP. The problem has been called the Lazy Bureaucrat Problem (LBP) (see, e.g. [11]), motivated by the special case in which $p_i = w_i$, for all $i \in I$. We deliberately refrain from this name since in this article we study a more general setting in which no correlation is assumed to exist between the item profits and item weights.

Similarly, in case of MPMKC, the goal is to maximize the profit of scheduled jobs while ensuring that a minimal possible subset of jobs exceeding the deadline is scheduled (i.e., to preserve a minimal cover). The MPMKC problem has been called the Greedy Boss Problem in [11], but again, since this name is also motivated by the setting $p_i = w_i$, for all $i \in I$, we will not use it in the remainder of this article.

Literature Overview The MCMKP problem has been introduced in [1] where it was shown that a more general problem variant with individual arrival times and deadlines is NP-hard. Following this seminal work, the problem has been extensively studied in the Computer Science literature and results concerning the problem complexity of special classes and approximation algorithms have been presented ([8, 11, 13]). On the contrary, concerning the development of exact methods, the problem has been tackled only in a recent preliminary study [7].

Depending on the coefficients p_i of the objective function, the following special cases are known in the (scheduling) literature:

- `min-number-of-jobs`: In this case, $p_i = 1$ for all $i \in I$ and the goal is to minimize the number of scheduled jobs. By reduction from the subset-sum problem, it has been shown in [8] that this problem variant is weakly NP-hard.

- `min-time-spent`: This variant corresponds to the problem in which $p_i = w_i$ for all $i \in I$. Since job profits are equal to job durations, this explains the motivation for calling the problem LBP, i.e., the goal of the "lazy employee" is to go home as early as possible, while having an excuse that there is no more time left to start a new job. This variant is also equivalent to the `min-makespan`, since all jobs arrive at the same time, and the goal is to minimize the time spent on executing jobs. In [11] two greedy heuristics and an FPTAS have been proposed for the `min-time-spent` objective with common arrival and deadlines.

- `min-weighted-sum`: This variant considers the most general objective function, with no restrictions on p_i values, and hence it covers all previous cases. This is the function considered in our paper. In a preliminary study [7], an $O(n^2C)$ dynamic programming procedure is proposed, thus showing that MCMKP is weakly NP-hard in this general setting. Three MIP formulations and their constraint programming (CP) counterparts are developed and tested, showing that the best performance is obtained by two of the proposed MIP models (these models are revised in Section 4).

Our Contribution This article aims at providing an in-depth study of exact methods (based on MIP- and DP-techniques) for solving MCMKP and MPMKC to provable optimality. Our contribution is as follows: First, we provide an efficient dynamic programming algorithm for the MCMKP that runs in $O(nC)$ time and $O(n+C)$ space. It improves upon the previous DP from [7] which was shown to run in $O(n^2C)$ time and $O(nC)$ space. Second, we provide new theoretical results concerning the strength of the two best performing MIP models from the literature. In an extensive computational study on a large and diverse set of benchmark instances, we demonstrate that the new DP algorithm outperforms the use of a state-of-the-art commercial MIP solver applied to these two best-performing models.

Finally, we prove that MCMKP and MPMKC are equivalent and that any optimal solution of the MPMKC can be obtained by an appropriate linear transformation. This implies that MPMKC is weakly NP-hard as well, and that there exists a DP procedure which runs in $O(n(W - C))$ time and requires $O(n + (W - C))$ space, where W is the sum of all item weights.

The remainder of this article is organized as follows: in the following section, we present structural properties of optimal MCMKP solutions that can be used to derive DP algorithms and MIP models. Section 3 is devoted to our new DP procedure for the MCMKP. In Section 4, we present and discuss new results for two MIP models for the MCMKP. In Section 5 we prove the equivalence between the MCMKP and the MPMKC. Computational results are provided in Section 6 and conclusions are drawn in Section 7.

2. Solution Properties for the MCMKP

In this section we point out some structural solution properties of MCMKP solutions that will be used in the remainder of this article.

In the following, we assume that all items are sorted in non-decreasing order according to their weight, i.e., $w_1 \leq w_2 \leq \cdots \leq w_n$. For each $i \in I$, let $C_i := C - w_i$, so we have $C_1 \geq C_2 \geq \ldots C_n$. We will also denote by $w_{\max} := \max_{i \in I} w_i (= w_n)$. Let $W := \sum_{i \in I} w_i$ and $P := \sum_{i \in I} p_i$. For a set $S \subset I$, let $w(S) = \sum_{i \in S} w_i$ and let $W_i = \sum_{j<i} w_j$.

It is not difficult to see that the following property is valid, due to the maximal packing property of an arbitrary MCMKP solution.

Property 1 ([7]). *The capacity used by an arbitrary feasible MCMKP solution $S \subset I$ is bounded from below as follows:*

$$w(S) \geq C - w_{\max} + 1.$$

In the following, we define a *critical item*, as an item whose weight gives a tight upper bound on the smallest possible item left out of any feasible solution.

Definition 5 (MCMKP-Critical Weight and MCMKP-Critical Item). *Denote by*

$$i_c = \arg\min\{i \in I \mid \sum_{j \leq i} w_j > C\}$$

the index of the critical item*, i.e., the index of the first item that exceeds the capacity, assuming all $i \leq i_c$ will be taken as well. The* critical weight*, denoted by w_c, is the weight of the critical item, i.e., $w_c = w_{i_c}$.*

Proposition 1 ([7]). *The weight of the smallest item left out of any feasible MCMKP solution is bounded from above by the critical weight w_c, i.e.:*

$$S \text{ is feasible } \Rightarrow \min_{i \notin S} w_i \leq w_c.$$

Consequently, the size of the knapsack can be bounded from below as:

$$w(S) \geq C - w_c + 1.$$

The validity of the above proposition follows from the simple observation: assume that there exists a feasible knapsack packing $S \subset I$ such that the weight of the smallest item left outside of S is $> w_c$. In that case, S must contain all items i such that $w_i \leq w_c$. By definition of w_c, the knapsack capacity occupied by S is then strictly greater than C, which is a contradiction.

Proposition 2. *Let $S \subset I$ be a feasible MCMKP solution. If item $i \in I$ is the smallest item left out of S, i.e., we have $S = \{1, \ldots, i-1\} \cup S'$ where $S' \subseteq \{i+1, \ldots, n\}$, the weight of S' can be bounded from below as:*

$$w(S') \geq C - \sum_{j<i} w_j - w_i + 1.$$

Proof. Assume that $w(S') \leq C - \sum_{j \leq i} w_j$. In that case, item i could be added to S, without exceeding the knapsack capacity, which contradicts the assumption that the packing S is maximal. □

Property 2. *Despite the minimization objective, imposing the upper bound on the size of the knapsack is not redundant, in general.*

In order to see that the latter property holds, consider the following example in which we are given three items such that $w_1 = 1$, $w_2 = 2$ and $w_3 = 3$, and $p_1 = 10$, $p_2 = p_3 = 1$, and let $C = 4$ be the knapsack capacity. Without imposing the upper bound on the size of the knapsack, the optimal solution will be to take the items $\{2, 3\}$ with the total cost of 2, whereas the optimal MCMKP solution is $\{1, 3\}$ with the total cost of 11.

In the `min-time-spent` case, however, imposing the upper bound is indeed redundant, as the next proposition shows.

Proposition 3. *If $w_i = p_i$ for all $i \in I$, the capacity of any optimal solution S will not exceed C, even without explicitly imposing this condition.*

Proof. Let us call the *relaxed problem* the MCMKP without the knapsack capacity constraint. Assume that $w_i = p_i$, for each $i \in I$, and that, on the contrary, S is an optimal solution (of minimal cardinality) of the relaxed problem, such that $w(S) > C$. Let k be the index of the least weight item in S, i.e., $k = \arg\min_{i \in S} w_i$. We will construct another solution S' as $S' = S \setminus \{k\}$. We have $w(S') + w_i \leq C$ for some item $i \in I \setminus S'$, since otherwise S' would be a maximal knapsack packing with a better objective value than S, which is a contradiction.
Denote by $\delta = C - w(S')$, which is the residual capacity of the knapsack that we obtain by removing the item k from S. We will now try to fill this gap by solving the following subproblem:

$$\max \sum_{i \in I \setminus S'} w_i x_i \quad \text{s.t.} \quad \sum_{i \in I \setminus S'} w_i x_i \leq \delta, \quad x_i \in \{0, 1\}.$$

Denote by S'' the optimal solution to this problem. Clearly $S'' \neq \emptyset$. Then, the new solution $\tilde{S} = S' \cup S''$ satisfies the following property: $w(\tilde{S}) \leq C$ and (by construction) $w(\tilde{S}) + w_i > C$ for all $i \notin \tilde{S}$, i.e., $\tilde{S}$ is another feasible solution of the relaxed problem with a better objective value than S, a contradiction. □

3. A New $O(nC)$ Dynamic Programming Algorithm for Solving the MCMKP

In this section we derive a new and effective Dynamic Programming (DP) algorithm for the MCMKP. The DP is based on the fact that, once the smallest item left out of the solution is known, MCMKP reduces to solving the knapsack problem with a lower and upper bound on its capacity (denoted LU-KP in the following). This property can be stated as follows:

Proposition 4. *The optimal MCMKP solution can be calculated as follows:*

$$\text{OPT} = \min_{i \in I : i \leq i_c} \left\{ KP(i, \underline{C}, \overline{C}) + \sum_{j<i} p_j \right\} \tag{2}$$

where $\underline{C} = C - W_i - w_i + 1$ *and* $\overline{C} = C - W_i$, *and*

$$KP(i, \underline{C}, \overline{C}) = \min\{\sum_{j>i} p_j y_j \ : \ \underline{C} \leq \sum_{j>i} w_j y_j \leq \overline{C}, y \text{ is binary}\}. \tag{3}$$

Proof. Let $\tilde{X}$ be an optimal MCMKP solution. Let item i be the smallest item taken out of $\tilde{X}$. Hence, all items from the set $\{1, \ldots, i-1\}$ belong to $\tilde{X}$. These items can be excluded from consideration, reducing the maximum knapsack capacity from C to $\overline{C} = C - W_i$. Then, the value of $\tilde{X}$ can be obtained as $KP(i, \underline{C}, \overline{C}) + \sum_{j<i} p_j$, where the LU-KP problem defined by (3) is solved on the remaining items from the set $\{i+1, \ldots, n\}$. Since solution $\tilde{X}$ has to be a maximal packing, the minimum capacity is defined as $\underline{C} = C - W_i - w_i + 1$ (see Property 2).

The optimal MCMKP solution value is found by trying all possible items $i \leq i_c$ and taking the best among the obtained solutions.

□

We now introduce the new DP algorithm which runs in $O(nC)$ time, and requires only $O(n + C)$ space. The pseudo-code of this new DP procedure is given in Algorithm 1 . A single (look-up) vector M of size C is used to keep track of the solution value for the problem $KP(i, \underline{C}, \overline{C})$. The update of values of M follows the DP procedure for solving a knapsack problem in which one searches for the subset of items whose total weight is exactly k ($k \in \{0, 1, \ldots, C\}$) such that the sum of profits of selected items is *minimized*. Accordingly, at each stage i ($i = n, \ldots, 1$), the value $M[k]$ stores the minimum profit of the subset of items that occupies exactly capacity k, while considering only the items from $\{i+1, \ldots, n\}$. Once the stage i is reached, such that $i \leq i_c$ (i.e., all items not larger than i can fit into the knapsack), the algorithm updates the best solution value according to formula (2).

During the execution of the algorithm a look-up table of size $O(nC)$ can also be built, with entries $M_i[k]$ storing the best LU-KP solution value considering items $\{i+1, \ldots, n\}$ and capacity k, $k \in \{0, \ldots, C\}$. A small example that illustrates our DP procedure (with the entries of matrix M) is given in Table 1.

Proposition 5. *The optimal solution value of the MCMKP can be computed by Algorithm 1 in $O(nC)$ time.*

Algorithm 1: An $O(nC)$ dynamic programming framework.

Data: Set of items I with profits p_i and weights w_i, $i \in I$. Knapsack capacity C.
Sums of profits $P := \sum_{i \in I} p_i$, $P_i = \sum_{j<i} p_i$ and weights $W := \sum_{i \in I} w_i$, $W_i := \sum_{j<i} w_i$
Result: Optimal MCMKP solution value OPT and required capacity C^*.

1 Set $M[0] := 0$, OPT $:= \infty$, $C^* := \infty$
2 **for** $k = 1, \ldots, C$ **do**
3   Set $M[k] := \infty$
4 **end**
5 **for** $i = n, \ldots, 1$ **do**
6   **if** $i \leq i_c$ **then**
7   Set $\overline{C} := \max\{0, C - W_i\}$, and $\underline{C} = \max\{0, C - W_i - w_i + 1\}$
8   Set $Tmp := \min_{\underline{C} \leq k \leq \overline{C}}\{M[k] + P_i\}$
9   **if** $Tmp <$ OPT **then**
10   Set OPT $:= Tmp$, $C^* := \arg\min_{\underline{C} \leq k \leq \overline{C}}\{M[k] + P_i\}$
11   **end**
12   **end**
13   **for** $k = C, \ldots, w_i$ **do**
14   Set $M[k] := \min\{M[k], M[k - w_i] + p_i\}$
15   **end**
16 **end**
17 **return** OPT *and* C^*

item	Capacity										
i	0	1	2	3	4	5	$\underline{C}$	$\overline{C}$	P_i	W_i	OPT
5	0	∞	∞	∞	1	∞	-	-	15	7	-
4	0	∞	∞	2	1	∞	0	1	13	4	13
3	0	∞	6	2	1	8	2	3	7	2	9
2	0	4	6	2	1	5	4	4	3	1	4
1	0	3	6	2	1	4	5	5	0	0	4

Table 1: Let the input instance be given as $|I| = 5$ with $w = (1, 1, 2, 3, 4)$ and $p = (3, 4, 6, 2, 1)$ and capacity $C = 5$. The index of the critical item is $i_c = 4$. The table represents the content of the vector M in the stages $i = 5, \ldots, 1$.

Proof. The key element of this new DP algorithm is the fact that the DP for the classical KP (see, e.g., [3, 17]) provides all necessary information to compute the optimal MCMKP solution value. While processing an item i (note that the items are processed in the backward direction, from the largest to the smallest one), we can compute the optimal values of the LU-KP considering an instance restricted to the subset of items $\{i+1, \ldots, n\}$ and for all $k \in \{0, \ldots, C\}$. This is done by implementing the following recursive procedure (cf. Line 14 in Algorithm 1):

$$M_i[k] := \begin{cases} \min\{M_i[k], M_{i+1}[k - w_i] + p_i\}, & \text{if } k \geq w_i \\ M_{i+1}[k], & \text{otherwise} \end{cases} \quad \forall i = n, \ldots, 1, k = 0, \ldots, C$$

where the initial values $M_{n+1}[k] := \infty$, for all $k = 1, \ldots, C$ and $M_{n+1}[0] := 0$.

Exploiting the relation between the MCMKP and the LU-KP solution values (cf. Proposition 5), in each iteration i (starting from an item i such that $i \le i_c$) we then obtain the optimal MCMKP solution value, assuming all unprocessed items (i.e., those from the set $\{1, \dots, i-1\}$) are taken in the solution. The value of this solution is stored in the variable Tmp (cf. Line 8), and the globally optimal solution is updated correspondingly. Observe that the items $\{1, \dots, i-1\}$ determine the remaining capacity of the knapsack (which is given as $\overline{C}$) that has to be filled using items from $\{i+1, \dots, n\}$, whereas $\underline{C}$ is determined by to the fact that the packing has to be maximal and that the item i is the smallest one taken out of the solution (cf. Property 2). This concludes the proof. $\square$

Observe that, so far, the Algorithm 1 only returns the optimal solution value OPT and the required knapsack capacity C^*. In order to recover the optimal solution S^*, there exist two possibilities. At the cost of increasing the space complexity from $O(n+C)$ to $O(nC)$, the optimal solution can be recovered in $O(n)$ time. To this end, it is sufficient to store the information whether the item i was added to the LU-KP knapsack in the current iteration or not, for each weight $k \in \{0, \dots, C\}$ and in each iteration $i = n, \dots, 1$. We keep track of the pointers $A_i[k] \in \{0, 1\}$ with the following meaning:

$$A_i[k] := \begin{cases} 1, & \text{if } M_i[k] = M_{i+1}[k - w_i] + p_i \\ 0, & \text{otherwise (i.e, if } M_i[k] = M_{i+1}[k]) \end{cases}, \quad \forall i = n, \dots, 1, \quad k = 0, \dots, C$$

After obtaining the optimal solution value OPT and the required capacity C^*, we can compute the optimal MCMKP solution S^* by applying a slight modification of the back-tracking procedure for the standard KP (for further details concerning this back-tracking see, e.g. Section 2.3 in [14]). In the following, we explain necessary modifications to this procedure. If $A_1[C^*] = 1$, starting from the cell $A_1[C^*]$, the back-tracking procedure is applied directly. If, otherwise, $A_1[C^*] = 0$, we first initialize $S^* = \{1, \dots, i^* - 1\}$, where i^* is the smallest index such that $A_{i^*}[C^*] = 1$. Starting from $A_{i^*}[C^*]$, we then continue enlarging S^* following the standard back-tracking procedure.

In the following we show that the optimal solution can be recovered, while keeping the space complexity $O(n+C)$.

Proposition 6. *The optimal solution and the optimal solution value of the MCMKP can be computed in $O(nC)$ time and $O(n+C)$ space.*

Proof. We refer to the recursive procedure for recovering the optimal solution of the classical KP described in Section 3.3 of [14]. This procedure runs in $O(nC)$ time and requires the four properties listed below to be satisfied by our problem. Let in the following OPT(I, k) denote the optimal MCMKP value for the instance (I, p, w, k) for any $k = 0, \dots, C$, and let $S^*(I, k)$ be the corresponding optimal solution. The properties are as follows:

1. There exists a procedure `Solve`(I, C) which computes for every integer $k = 0, \dots, C$ the value OPT(I, k) and requires $O(nC)$ time and $O(n+C)$ space.

2. If $n = 1$, then `Solve`(I, C) also computes the optimal solution $S^*(I, k)$ for all $k = 0, \dots, C$.

3. For every partitioning I_1, I_2 of I, there exist capacities C_1, C_2 such that $C_1 + C_2 = C$ and OPT$(I_1, C_1) \cup$ OPT$(I_2, C_2) =$ OPT(I, C), and

4. For every partitioning I_1, I_2 of I with $|I_2| = 1$, and arbitrary capacities C_1, C_2, there exists a procedure `Merge` that combines $\text{OPT}(I_1, C_1)$ and $\text{OPT}(I_2, C_2)$ into $\text{OPT}(I_1 \cup I_2, C_1 + C_2)$ in $O(C_1)$ time.

Regarding the first property, observe that a minor modification of the proposed Algorithm 1 is required, in which we keep track of the optimal MCMKP solution value for all $k = 0, \ldots, C$, and not only for $k = C$. More precisely, for each $k = 0, \ldots, C$, one has to keep track of $\underline{C}$, $\overline{C}$ and OPT, which means that the space complexity increases from $C + n$ to $4C + n$.

The remaining properties are trivially satisfied. Hence, by recursively calling the `Solve` procedure over an arbitrary chosen non-trivial partitioning (I_1, I_2) of I, one can rebuild the optimal solution $S^*(I, C)$ in $O(nC)$ time. □

4. MIP Models for the MCMKP

Several MIP and constraint programming (CP) formulations for the MCMKP have been proposed in [7] where it has been demonstrated that CP formulations are not competitive against MIP models. In the following, we first recapitulate the two best performing MIP formulations among them, and then we derive new theoretical results concerning the strength of their LP-relaxations. The results are derived using the LP-duality theory and the theory of Benders decomposition.

4.1 An ILP Formulation for the MCMKP in the Natural Variable Space

In the following let the binary variables x_i be set to one if the item i is selected, and to zero, otherwise. Each feasible solution has to fit into the knapsack, but it also has to ensure the maximal packing property (i.e., adding an arbitrary additional item left outside of the solution, results in exceeding the knapsack capacity). The formulation given below exploits the Proposition 1, and considers only the items left outside of S whose weight does not exceed the critical weight. Let $C_c = C - w_c$. The model reads as follows:

$$(F) \qquad \min \sum_{i \in I} p_i x_i \tag{4}$$

$$\sum_{i \in I} w_i x_i \leq C \tag{5}$$

$$\sum_{j \in I, j \neq i} w_j x_j + w_i(1 - x_i) \geq (C + 1)(1 - x_i) \qquad \forall i \in I : i \leq i_c \tag{6}$$

$$\sum_{j \in I} w_j x_j \geq C_c + 1 \tag{7}$$

$$x_i \in \{0, 1\} \qquad \forall i \in I \tag{8}$$

Constraint (5) is a knapsack constraint stating that the weight of all selected items cannot exceed the available capacity C. Inequalities (6) are based on Proposition 1 and Property 2 – they make sure that adding each additional item i such that $i \leq i_c$ will exceed the total capacity. We will refer to them as *covering inequalities* associated to items $i \leq i_c$. Finally, the *global covering constraint* is derived from the global lower bound given in Proposition 1. The constraint is redundant in the ILP formulation, but it improves the quality of LP-bounds (see [7]).

Furthermore, coefficients of the covering constraints (6) and (7) can be down-lifted as shown in the following Proposition.

Proposition 7 ([7]). *For any $i \in I$, $i \leq i_c$, the associated covering inequalities* (6) *can be down-lifted to:*

$$\sum_{j \in I, j \neq i} \min\{w_j, C_i + 1\} x_j + \min\{w_c, C_i + 1\} x_i \geq C_i + 1 \tag{9}$$

Similarly, coefficients of the constraint (7) *can be down-lifted to* $\min\{w_j, C_c + 1\}$*, for all* $j \in I$*.*

Observe that after this lifting, the covering inequality (6) associated to i_c and the global covering inequality (7) become identical. In the following, we will denote by (F_l) the *lifted formulation* (*F*), in which constraints (6) and (7) are replaced by their lifted counterparts as described in Proposition 7.

4.2 An Extended Formulation

Recall that for any knapsack solution to be feasible for the MCMKP it is sufficient that adding the weight of the *smallest item* left out of the knapsack already violates the capacity C. Our next model encodes this information by extending the previous model with a single non-negative continuous variable z modeling the weight of a smallest item that is left out of the solution. The model reads as follows:

$$(\mathrm{F}_e) \qquad \min \sum_{i \in I} p_i x_i$$

$$\sum_{i \in I} w_i x_i \leq C \tag{10}$$

$$\sum_{i \in I} w_i x_i + z \geq C + 1 \tag{11}$$

$$z \leq w_c - (w_c - w_i)(1 - x_i) \qquad \forall i \in I, i \leq i_c \tag{12}$$

$$x_i \in \{0, 1\} \qquad \forall i \in I \tag{13}$$

$$z \geq 0 \tag{14}$$

Validity of this model can be easily verified. Coupling constraints (12) make sure that $z \geq 0$ is at most the weight of the smallest item not included in the solution. Due to the covering constraint (11) and the minimization objective function (with $p_i \geq 0$), it follows that z will be exactly the weight of the smallest item left outside. For the same reasons as above (cf. Proposition 1), it is not necessary to impose constraints (12) for $i > i_c$.

Compared to the model (F) from above, this model contains only a single additional variable, but, thanks to the reformulation, the structure of the constraint matrix is significantly simplified. Besides the packing constraint (10) and the covering constraint (11), the remaining matrix has a diagonal structure, plus one column of ones (corresponding to the variable z). Therefore, it is expected that the general-purpose MIP solvers can be even more efficient when solving (F_e) than the formulation (F).

We now derive an interesting connection between (F_e) and (F). It turns out that by applying the Benders decomposition to (F_e) and projecting out the z variable, we end up with two very similar models.

Proposition 8. *Projecting out z variables from (F_e) results in the formulation (F) in which constraints* (6) *are down-lifted as follows:*

$$\sum_{j \in I, j \neq i} w_j x_j + w_c x_i \geq C_i + 1$$

Proof. After eliminating all constraints involving z variable from the (F_e), the *reduced* master problem boils down to the classical KP, but with the minimization objective:

$$\min\{p^T x \mid w^T x \leq C, x \text{ is binary }\} \tag{15}$$

A solution $\bar{x}$ of the master problem (15) is feasible if and only if we can find z that satisfies the following constraints:

$$z \geq C + 1 - \sum_{j \in I} w_j \bar{x}_j \qquad (\gamma) \tag{16}$$

$$-z \geq -w_i - (w_c - w_i)\bar{x}_i \quad \forall i \in I : i \leq i_c \qquad (\alpha_i) \tag{17}$$

$$z \geq 0 \tag{18}$$

Farkas lemma states that a linear system of inequalities $\{Az \geq b,\ z \geq 0\}$ is feasible if and only if $u^T b \leq 0$ for all $u \geq 0$ such that $u^T A \leq 0$. Let $\gamma \geq 0$ be a dual variable associated to constraint (16) and let $\alpha_i \geq 0$, for all $i \leq i_c$ be dual variables associated to (17). By Farkas lemma, $\bar{x}$ is feasible if and only if the optimal solution value of the following system is non-positive:

$$\max \gamma(C + 1 - \sum_{j \in I} w_j \bar{x}_j) + \sum_{i \leq i_c} \alpha_i(-w_i - (w_c - w_i)\bar{x}_i) \tag{19}$$

$$\text{s.t.} \ -\sum_{i \leq i_c} \alpha_i + \gamma \leq 0 \tag{20}$$

$$\alpha, \gamma \geq 0 \tag{21}$$

We observe that the optimal solution of the latter problem satisfies $\gamma = \sum_{i \leq i_c} \alpha_i$, since $\bar{x}$ is a feasible master solution and hence it satisfies the knapsack capacity constraint, so we have $C + 1 - \sum_i w_i \bar{x}_i > 0$. The dual problem now boils down to:

$$\max_{\alpha \geq 0} \quad \sum_{i \leq i_c} \alpha_i \left(C + 1 - \sum_{j \in I} w_j \bar{x}_j - w_i - (w_c - w_i)\bar{x}_i \right)$$

This problem is unbounded (i.e., $\bar{x}$ is infeasible) if and only if there exists $i \in I$, $i \leq i_c$, such that

$$C + 1 - \sum_{j \in I, j \neq i} w_j \bar{x}_j - w_c \bar{x}_i - w_i > 0$$

Hence, all infeasible solutions $\bar{x}$ of the master problem can be cut-off by the following (compact) family of feasibility constraints:

$$\sum_{j \in I, j \neq i} w_j x_j + w_c x_i \geq C + 1 - w_i, \quad \forall i \in I : i \leq i_c,$$

what concludes the proof. $\square$

Corollary 1. *The value of the LP-relaxation of (F_e) and of the following model are the same:*

$$\min \sum_{i \in I} p_i x_i$$

$$\sum_{i \in I} w_i x_i \leq C \tag{22}$$

$$\sum_{j \in I, j \neq i} w_j x_j + w_c x_i \geq C_i + 1 \qquad \forall i \leq i_c \tag{23}$$

$$x_i \in \{0, 1\} \qquad \forall i \in I$$

Corollary 2. *If $\min\{w_j, C_c + 1\} = w_j$, for all $j \in I$, then the LP-relaxation bounds of (F_l) and (F_e) are the same.*

Corollary 3. *In general, LP-relaxation bounds of (F_l) can be stronger than those of (F_e).*

In order to see that this is the case, consider the same example given in Table 1. Inequalities (9) and (23) are identical for all $i < i_c$, however for $i = i_c = 4$, the inequality (9) in the model (F_l) is $x_1 + x_2 + 2x_3 + 3x_4 + 3x_5 \geq 3$, whereas the associated inequality (23) is weaker and reads $x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 \geq 3$.

5. Equivalence Between MCMKP and MPMKC

In this section we prove the following result:

Proposition 9. *Any feasible solution S of the MCMKP on an instance (I, p, w, C) corresponds to a feasible solution $S' = I \setminus S$ of the MPMKC on instance (I, p, w, C') where $C' = W - C$.*

Before showing this result, we will discuss the critical item of the MPMKC and a counterpart of the extended formulation for the MPMKC. As above, we assume that items are sorted in non-decreasing order according to their weights. Observe that the index of the critical item and the corresponding weight for the MPMKC are calculated differently.

Definition 6 (MPMKC-Critical Weight and MPMKC-Critical Item)**.** *Denote by*

$$\bar{i}_c = \arg\max\{i \in I \mid \sum_{j \geq i} w_j \geq C\}$$

the index of a MPMKC-critical item*, i.e., the index of the first item that covers the knapsack, assuming all $i \geq \bar{i}_c$ will be taken as well. The* MPMKC-critical weight*, denoted by $\bar{w}_c$, is the weight of the critical item, i.e., $\bar{w}_c = w_{\bar{i}_c}$.*

Proposition 10. *The weight of the smallest item taken into any feasible MPMKC solution is bounded from above by the critical weight $\bar{w}_c$, i.e.:*

$$S \textit{ is feasible } \Rightarrow \min_{i \in S} w_i \leq \bar{w}_c.$$

Consequently, the capacity used by any minimal knapsack cover is bounded from above as:

$$w(S) \leq C + \bar{w}_c - 1.$$

Proof. Suppose S is a feasible MPMKC solution such that $\min_{i \in S} w_i > \bar{w}_c$. Hence, S contains only items i such that $i > \bar{i}_c$. However, by definition of the critical item, it follows that $\sum_{i: i > \bar{i}_c} w_i \leq C - 1$ which means that S is not a cover, a contradiction.

For the second part of the proposition, observe that capacity used by any feasible solution can be bounded from above as $C - 1 + w_{\max}$. However, knowing that removal of even the smallest item from this solution results in capacity $\leq C - 1$, leads to the conclusion that $w(S) \leq C - 1 + \bar{w}_c$. □

We can now prove the following lemma:

Lemma 1. *We have that the index of a critical item i_c for the MCMKP on instance (I, p, w, C) and the index of a critical item $\bar{i}_c$ for the MPMKC on instance (I, p, w, C') are the same, i.e., it holds:*

$$\arg\min\{i \in I \mid \sum_{j \leq i} w_j > C\} = \arg\max\{i \in I \mid \sum_{j \geq i} w_j \geq W - C\}$$

Proof. Observe that $\sum_{j \leq i} w_j$ can be written as $W - \sum_{j > i} w_j$ so that i_c is the smallest index of an item i such that $\sum_{j > i} w_j < W - C$. Hence, for i_c it must hold: $\sum_{j \geq i_c} w_j \geq W - C$. Obviously, for any $i < i_c$, we have $\sum_{j \geq i} w_j \geq \sum_{j \geq i_c} w_j \geq W - C$. So, we conclude that i_c is the maximal index of an item that satisfies the latter property, which exactly corresponds to the definition of the index of a critical item $\bar{i}_c$ for the instance (I, p, w, C') for the MPMKC. □

Extended Formulation for the MPMKC Let binary variables y_i be set to one if item i is taken into MPMKC solution, and to zero, otherwise. The following (extended) MIP formulation is a valid model for the MPMKC, in which the variable $\tilde{z}$ takes the weight of the minimum item in the solution.

$$\max \sum_{i \in I} p_i y_i \tag{24}$$

$$\sum_{i \in I} w_i y_i \geq C \tag{25}$$

$$\sum_{i \in I} w_i y_i - \tilde{z} \leq C - 1 \tag{26}$$

$$\tilde{z} \leq w_i y_i + \bar{w}_c (1 - y_i) \qquad \forall i \in I : i \leq \bar{i}_c \tag{27}$$

$$y \in \{0, 1\} \qquad \forall i \in I \tag{28}$$

$$\tilde{z} \geq 0 \tag{29}$$

Constraint (25) ensures that the solution is a knapsack cover, whereas constraint (26) makes sure that this cover is minimal, i.e., whenever the smallest item is taken out of the solution, the weight of the remaining selected items remains $\leq C-1$. Finally, constraints (27) ensure that $\tilde{z}$ takes the value of the smallest item in the solution, by exploiting the property of Proposition 10.

Proof of Proposition 9. The linear transformation $y_i = 1 - x_i$, for all $i \in I$, allows to translate the extended formulation (F_e) for the MCMKP into the extended formulation (24)-(29) for the MPMKC and vice versa. We start with the model (F_e) given in Section 4.2, and apply this linear transformation. The model (F_e) translates into the following one:

$$P + \min \sum_{i \in I} (-p_i) y_i$$

$$\sum_{i \in I} w_i y_i \geq C' \tag{30}$$

$$\sum_{j \in I} w_j y_j - z \leq C' - 1 \tag{31}$$

$$z \leq w_c - (w_c - w_i) y_i \qquad \forall i \in I : i \leq i_c \tag{32}$$

$$y_i \in \{0, 1\} \qquad \forall i \in I$$

Observing that the minimization can be turned into maximization and, using the facts that $C' = C - W$ and $i_c = \bar{i}_c$ (see Lemma 1), it immediately follows that the transformed model and the model (24)-(29) are the same for the instance (I, p, w, C'), which concludes the proof. □

Hence, in order to solve MPMKC on the instance (I, p, w, C), it is sufficient to solve the complementary MCMKP on the instance $(I, p, w, W - C)$ and take the items left out of the optimal MCMKP solution. The above result together with Proposition 5 implies the following:

Corollary 4. *The problem MPMKC on an instance (I, p, w, C) is weakly NP-hard and there exists a DP algorithm that solves it in $O(n(W - C))$ time.*

6. Computational Study

The primary goal of this computational study is to compare the performances of the Dynamic Programming (DP) described in Section 3 with the MIP models discussed in Section 4. To the best of our knowledge, these models are the best formulations proposed so far in literature. The computational results presented in [7] allow us to conclude that both models, i.e., (F_l) and (F_e), outperform in terms of computational performance the DP algorithm proposed in [7]. Recall that this latter DP algorithm runs in $O(n^2 C)$ time. As a matter of fact, it already struggles to tackle instances with 100 items and it is effective only in solving small size instances (see [7] for more details). For this reason, we decided to exclude it from the this computational study.

The experiments have been performed on a computer with a 2.5 Ghz Intel Xeon CPU processor and 64GB RAM, running a 64 bits Linux operating system. All the codes were compiled with `g++` (version 4.8.4) using `-O3` as optimization option. We have tested the MIP models using `Cplex 12.6.3`, single-threaded (called for brevity `Cplex` in the following). All `Cplex` parameters are at their default values except the optimality tolerances which have been set to 0 in order to be able to compare the solutions of the models with the solutions of the DP algorithm. In all our tests we imposed a time limit of 1 000 seconds and memory limit of 3GB.

A second goal of this computational study is also to determine the size of the instances which can be solved to proven optimality within short running time. In [7], a very diverse banchmark set of instances with up to 2 000 items has been proposed. In this manuscript we consider the same classes of instances, but, in order determine the limits of the proposed exact methods, we also introduce larger instances with up to 5 000 items. In the following the test bed of instances is described in details.

6.1 Benchmark Instances

Nine classes of MCMKP instances have been randomly generated by defining weights and profits following the procedure for the classical KP described in [15] (u.r. stands for "uniformly random integer"). The instance generator is available at [20].

1. *Uncorrelated*: w_j u.r. in $[1, \overline{R}]$, p_j u.r. in $[1, \overline{R}]$.
2. *Weakly correlated*: w_j u.r. in $[1, \overline{R}]$, p_j u.r. in $[w_j - \overline{R}/10, w_j + \overline{R}/10]$ so that $p_j \geq 1$.
3. *Strongly correlated*: w_j u.r. in $[1, \overline{R}]$, $p_j = w_j + \overline{R}/10$.
4. *Inverse strongly correlated*: p_j u.r. in $[1, \overline{R}]$, $w_j = p_j + \overline{R}/10$.
5. *Almost strongly correlated*: w_j u.r. in $[1, \overline{R}]$, p_j u.r. in $[w_j + \overline{R}/10 - \overline{R}/500, w_j + \overline{R}/10 + \overline{R}/500]$.
6. *Subset-sum*: w_j u.r. in $[1, \overline{R}]$, $p_j = w_j$.
7. *Even-odd subset-sum*: w_j even value u.r. in $[1, \overline{R}]$, $p_j = w_j$, c odd.
8. *Even-odd strongly correlated:* w_j even value u.r. in $[1, \overline{R}]$, $p_j = w_j + \overline{R}/10$, c odd.
9. *Uncorrelated with similar weights:* w_j u.r. in $[100\,\overline{R}, 100\,\overline{R} + \overline{R}/10]$, p_j u.r. in $[1, \overline{R}]$.

For each instance class and value of $\overline{R} \in \{1000, 10000\}$, we generated 30 MCMKP instances by considering all combinations of (i) number of items $n \in \{10, 20, 30, 40, 50, 100, 500, 1000, 2000, 5000\}$; (ii) capacity $C \in \{\lfloor 25\% W \rfloor, \lfloor 50\% W \rfloor, \lfloor 75\% W \rfloor\}$ (and C increased by 1, if even, for classes 7 and 8); thus obtaining 540 MCMKP instances.

6.2 Initialization and Primal Heuristics

Thanks to extensive preliminary tests we observed that `Cplex` sometimes struggles to find good primal solutions for models (F_l) and (F_e). Accordingly we decided to implement Initialization and Primal Heuristics. The initialization heuristic is the *greedy-comb* proposed in [7]. The approach consists of two basic steps:

(i) examining the items according to a pre-specified order,

(ii) adding the current item to the solution iff its weight does not exceed the current residual capacity.

Six different orderings are incorporated in this *greedy-comb* procedure, and the best solution found is returned. The procedure *greedy-comb* is very fast and produces high quality initial solutions, with gaps to the optimal solutions being between 1% and 6% on our benchmark set (see [7] for further details).

In addition to this Initialization Heuristic, we apply a new and effective Primal Heuristic that can be executed within the nodes of the branch-and-cut scheme of `Cplex`. Once a LP-solution $\bar{x}$ is calculated, we call *greedy-comb* by weighting the item scores using the values of $\bar{x}_i$, for all $i \in I$. Then the items are resorted according to the new score values, and *greedy-comb* is then executed. This primal heuristic produces effective speedups as it is shown in the next Sections.

6.3 Quality of the LP Relaxation of the Models

Table 2 shows the gaps between the optimal solution value and the value of the Linear Programming (LP) relaxations of (F_l) and (F_e). The table is structured according to the instance categories, i.e., each entry reports the average gap of the 60 instances of the considered category. The gaps are calculated as $(z^* - z_R)/z^* 100$, where z^* is the optimal solution value and z_R is the optimal value of the LP relaxation. We observe that the gaps of both formulations are very similar, for categories 1, 2, 4, 5 and 9 the average gaps coincide. The instance of category 1 are characterized by the largest average gaps (12.30%). Aside from this category and category 9 with an average gap of 5.98%, the gaps for all other categories are under 3%.

Table 2: Average percentage LP gaps for (F_l) and (F_e) .

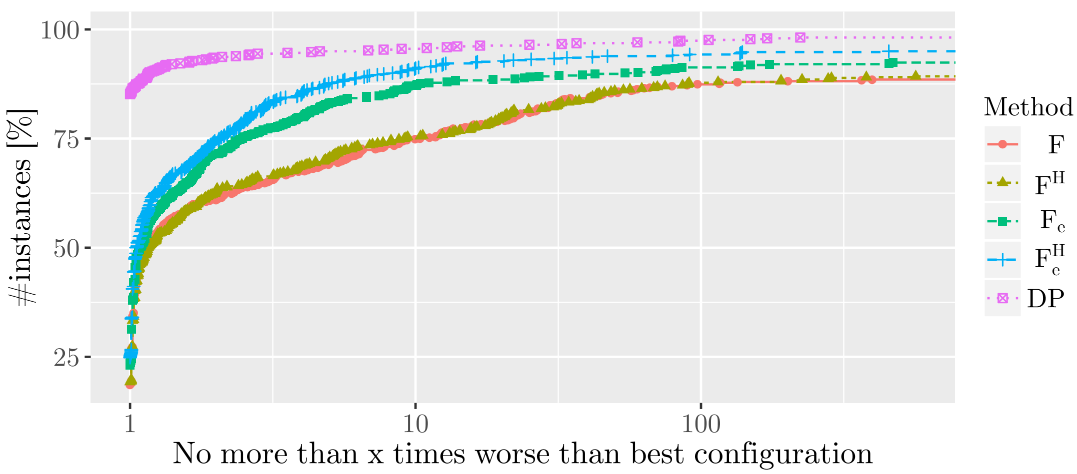
	instance category								
	1	2	3	4	5	6	7	8	9
(F_l)	12.30	2.87	2.47	2.27	2.41	2.63	2.64	2.56	5.98
(F_e)	12.30	2.87	2.51	2.27	2.41	2.65	2.66	2.59	5.98

6.4 Comparison Between the Proposed Exact Methods

The performances of the DP algorithm, the (F_l) and the (F_e) are compared. For the two MIP models additional configurations are considered, in which the basic models are enhanced by our Initial and Primal Heuristics (denoted by $(\mathrm{F}_l)^H$ and $(\mathrm{F}_e)^H$). Instances not solved to proven optimality within the timelimit of 1 000 seconds, or which have terminated due to the memorylimit, are counted as 1 000 seconds in the following results.

In order to give a graphical representation of the relative performance of the proposed approaches, we report a performance profile in Figure 1. For each instance we compute a normalized time τ as the ratio of the computing time of the considered solution method over the minimum computing time for solving the instance to optimality. For each value of τ in the horizontal axis, the vertical axis reports the percentage of the instances for which the corresponding method spent at most τ times the computing time of the fastest method. It shows that *DP* outperforms the MIP models both in terms of speed and number of solved instances. In about 85% of the instances the DP algorithm is the fastest one and it is capable to solve almost 99% of the instances to proven optimality. The second best approach is $(\mathrm{F}_e)^H$ followed by (F_e). The models $(\mathrm{F}_l)^H$ and (F_l) have similar poor performances with respect to the DP algorithm. In the following, we will take a more detailed look at the results.

Figure 1: Performance profile of the different approaches.



An aggregated picture of our computational results is given in Table 3. The table reports the average runtime to solve to proven optimality all instances for a given number of items (n) (average values over 54 instances are reported in each line of the table). We can conclude that instances with items up to 100 items can be solved within a few seconds by all the proposed approaches. Starting with the instances having 500 or more items, the difficulty significantly increases, and the table computationally proves that the DP algorithm globally outperforms both MIP-approaches for all classes of instances starting from $n = 30$ (with speedup factors ranging between 2 and 10).

As far as the effect of using Initialization and Primal Heuristics is concerned, the tables show that a sensible speed up can be obtained in most of the cases (excluding the trivial instances with less than 100 items). The effect is more marked up for (F_e) and 2000 items where the use of heuristics nearly halve the runtime. Interestingly, for (F_e), the number of items does not seem to influence the runtime too much, as for 1000 items, the average runtime is actually slightly faster than for 500 items. On the other hand, as expected, for DP the runtime grows, as the number of items grows.

Table 3: Performance comparison in term of average computing time by varying the number of items.

n	#	(F_l)	$(\mathrm{F}_l)^H$	(F_e)	$(\mathrm{F}_e)^H$	DP
10	54	0.00	0.00	0.00	0.00	0.00
20	54	0.01	0.02	0.02	0.02	0.02
30	54	0.07	0.11	0.12	0.13	0.04
40	54	0.17	0.35	0.39	0.41	0.07
50	54	0.20	0.35	0.21	0.21	0.10
100	54	1.22	13.16	13.23	5.83	0.42
500	54	164.15	178.57	171.25	145.85	10.72
1000	54	145.29	152.99	128.81	104.30	48.75
2000	54	333.35	317.16	292.83	159.67	76.86
5000	54	828.87	770.35	389.12	244.81	149.93

Table 4 reports the number of instances not solved to proven optimality within the given timelimit (1 000 seconds) and memorylimit (3 GB). The structure of this table is the same as for Table 3, i.e., each line corresponds to a subset of 54 instances grouped by the number of items (n). We see that for (F_l) and $(\mathrm{F}_l)^H$ struggle to solve instances starting from $n = 500$. For the largest instances with $n = 5\,000$, 41 and 37 instances remain unsolved by (F_l) and $(\mathrm{F}_l)^H$, respectively. Also (F_e) and $(\mathrm{F}_e)^H$ struggle to solve instances starting from $n = 500$ but more instances could be solved, i.e, only 17 and 10 instances remain unsolved by (F_e) and $(\mathrm{F}_e)^H$, respectively. A possible explanation is the density of the model (F_l), that causes the memorylimit to be reached more often that for the model (F_l) (which possesses an almost diagonal structure of the constraint matrix). We therefore conclude that model (F_e) outperforms model (F_l) in terms of the number of instances solved to proven optimality. Table 4 also demonstrates the beneficial effect of our heuristic algorithms: globally, $(\mathrm{F}_l)^H$ solves 4 more instance than (F_l) and $(\mathrm{F}_e)^H$ solves 14 more instances than (F_e).

As far as the performances of the DP are concerned, the DP algorithm only starts struggling to solve instances from $n = 1\,000$ and only 10 out of 540 instances remains unsolved by this exact approach. As a comparison, model $(\mathrm{F}_l)^H$ fails in total in 58 instances while model $(\mathrm{F}_l)^H$ fails in 27 instances. This allows us to conclude that our new DP algorithm computationally outperforms both models $(\mathrm{F}_l)^H$ and $(\mathrm{F}_e)^H$ also for the total number of instances solved to proven optimality.

Table 4: Instances unsolved within timelimit and memorylimit varying the number of items.

n	#	(F_l)	$(\mathrm{F}_l)^H$	(F_e)	$(\mathrm{F}_e)^H$	DP
10	54	0	0	0	0	0
20	54	0	0	0	0	0
30	54	0	0	0	0	0
40	54	0	0	0	0	0
50	54	0	0	0	0	0
100	54	0	0	0	0	0
500	54	8	9	9	7	0
1000	54	5	5	6	5	2
2000	54	8	7	9	5	3
5000	54	41	37	17	10	5

It is worth mentioning that all instances which remain unsolved for DP are of category 9, which has much larger weights compared to the other instance classes. This class of instances is specifically conceived to create instances which are hard for DP algorithms. In order to have a clear picture on the performances of the proposed exact methods on the different class of instances we report Table 5. Instances terminated due to memorylimit are counted as taking 1 000 seconds as for the other tables. The table shows the average runtime to solve to proven optimality the instances grouped by category. Each column reports averages values for 60 instances and on the rows the different exact methods are given.

Table 5: Performance comparison in term of average computing time grouping istances by category.

	instance category								
	1	2	3	4	5	6	7	8	9
#	60	60	60	60	60	60	60	60	60
(F_l)	115.47	97.85	113.94	287.77	213.64	164.23	149.37	98.06	86.94
$(\mathrm{F}_l)^H$	119.10	80.40	99.42	293.30	211.08	157.00	166.98	73.05	90.56
(F_e)	17.25	11.14	223.68	270.88	111.09	56.06	23.10	154.65	28.53
$(\mathrm{F}_e)^H$	17.82	8.73	92.56	240.61	99.01	45.54	13.52	46.49	30.82
DP	5.72	5.85	5.49	6.89	5.45	5.34	5.44	5.47	212.57

For both MIP-approaches (regardless of using the heuristics or not), instance category 4 is the most difficult one, while at least for (F_e) (and $(\mathrm{F}_e)^H$), instances of categories 1 and 2 are rather easy. For all categories except category 9, the DP algorithm clearly outperforms the MIP-approaches. As previously mentioned, the instances of category 9 have much larger coefficients which leads to a higher value of C and accordingly the performances of the DP algorithm deteriorates. For the instances of category 9, (F_e) tends to be the best approach. Finally, we can conclude that (F_e) outperforms (F_l) for all categories of instances.

In Tables 6 ($\overline{R} = 1000$) and 7 ($\overline{R} = 10000$) we show the runtime to optimality for instances with at least 100 items in detail. We report the runtime for $(\mathrm{F}_l)^H$, $(\mathrm{F}_e)^H$ and DP. Observe that the instances that were not solved to optimality by DP, were solved by $(\mathrm{F}_e)^H$, and vice versa, so that we are able to report optimal solution values for the whole benchmark set. These values are shown in columns z^*.

Summary This computational study computationally demonstrates that the DP algorithm outperforms the MIP-based approaches for 8 out of 9 instance categories. The only exception are the instances of category 9. Recall that category 9 is built with about 100 times larger weights than the remaining categories. Since C is defined as a percentage of the sum of all item weights, and the time complexity of DP is $O(nC)$, this easily explains the relatively poor performance of DP on this instance category. On the contrary, for this same category, $(\mathrm{F}_e)^H$ seems to be the best performing method. For the DP algorithm, Tables 6 and 7 confirm that the runtime grows with n and with C. For all approaches, instances with $\overline{R} = 1000$ tend to be easier than the ones with $\overline{R} = 10000$, this fact can be explained by smaller values of item weights. No clear picture emerges instead regarding the dependence of the difficulty on the capacity C for the MIP-based approaches, e.g., $(\mathrm{F}_e)^H$ takes 239 seconds for $1000 - 4 - 2000 - 0.50$ ($\overline{R} - cat - n - C$), against 2 seconds for $1000 - 4 - 2000 - 0.25$ and 8 seconds for $1000 - 4 - 2000 - 0.25$, while for $1000 - 4 - 500$ the results are reversed, with both $C = 25\%$ and $C = 75\%$ not being solved within the timelimit, while $C = 50\%$ takes 8 seconds. The most difficult instance category for the MIP-based approaches is category 4, for $\overline{R} = 10000$ very few instances can be solved to optimality. On the other hand, the DP algorithm requires at most 155 seconds for these instances. Aside from the difficulty with instances of category 9, the DP algorithm has a similar computational behaviour in all the other classes of instances.

7. Conclusions and Future Work

Problems which impose inclusion-wise maximality of a feasible solution while at the same time minimizing a non-negative cost function, have recently received much attention from the Theoretical Computer Science community. In particular, the so-called Lazy Bureaucrat Problem has been studied from approximation and complexity points of view (see e.g, [11] and the references therein).

In this article we studied two versions of this problem denoted by Minimum-Cost Maximal Knapsack Packing (MCMKP) and the Maximum-Profit Minimal Knapsack Cover (MPMKC). These are two problems closely related to the knapsack problem, with very important applications in scheduling. In this manuscript we proposed an exact algorithm based on dynamic programming and two effective MIP formulations. Our methods allowed us to solve to proven optimality instances with up to 5 000 items in short computational time.

Concerning future work, it is worth mentioning that many related "lazy" problems that have recently been introduced in the Computer Science literature (see, e.g., [6, 9] for the lazy bin packing problem, [5] for the minimum maximal independent set problem, [4] for the max-min vertex cover problem, [2] for maximum cardinality minimal dominating set problem, or [12] for the lazy matroid problem) remain quite unexplored

Table 6: Runtime in seconds and best solution value for each instance with $\overline{R} = 1000$.

		$(\mathrm{F}_l)^H$			$(\mathrm{F}_e)^H$			DP			z^*		
cat.	$n \diagdown C$	25%	50%	75%	25%	50%	75%	25%	50%	75%	25%	50%	75%
1	100	0.24	0.38	0.46	0.40	0.11	0.33	0.00	0.00	0.00	2396	9418	20420
	500	1.27	1.15	4.98	0.85	0.25	1.45	0.04	0.08	0.12	11034	43945	95648
	1000	3.14	13.10	0.90	1.56	1.13	0.27	0.16	0.33	0.48	22225	90721	199140
	2000	50.59	49.23	126.22	7.75	6.47	7.65	0.64	1.29	1.94	47146	189380	417695
	5000	76.12	359.79	ML	20.69	17.28	17.15	3.97	8.91	11.88	115800	470676	1058114
2	100	0.36	0.36	0.73	0.22	0.16	0.52	0.00	0.00	0.00	10246	22062	34855
	500	0.63	4.62	5.75	0.41	0.69	2.44	0.04	0.09	0.13	49973	107440	169917
	1000	8.00	4.94	1.22	1.93	1.26	0.30	0.17	0.33	0.50	100823	216924	342464
	2000	20.99	47.40	25.50	3.99	5.92	6.01	0.67	1.32	2.03	206505	444553	701755
	5000	68.01	45.38	ML	13.93	6.76	17.99	4.10	8.29	12.52	524047	1122836	1768368
3	100	0.05	0.13	0.41	0.15	0.04	0.02	0.00	0.00	0.00	13991	28182	42873
	500	0.58	7.73	1.09	0.12	0.44	0.32	0.03	0.07	0.11	68609	138518	210727
	1000	1.18	1.56	6.49	0.51	1.10	1.77	0.14	0.29	0.44	133599	269798	410797
	2000	3.22	6.32	5.40	1.58	3.07	1.52	0.59	1.18	1.78	269059	543419	827379
	5000	32.86	ML	ML	18.97	40.46	12.24	3.79	8.59	12.40	683746	1379993	2100739
4	100	0.04	0.37	0.19	0.05	0.58	0.10	0.00	0.00	0.00	10117	22852	35951
	500	TL	3.20	TL	TL	8.73	TL	0.05	0.10	0.15	51203	114165	179505
	1000	10.02	28.87	6.86	2.12	12.81	1.86	0.19	0.38	0.57	99064	221845	349770
	2000	5.82	TL	140.83	2.13	230.83	7.52	0.77	1.54	2.31	200321	448155	706438
	5000	363.73	ML	ML	TL	19.96	41.41	4.88	10.78	15.96	512400	1142721	1799085
5	100	0.75	0.11	0.37	0.10	0.01	0.18	0.00	0.00	0.00	13744	27599	42046
	500	TL	1.16	1.66	467.47	0.26	0.32	0.03	0.07	0.11	66739	134695	205247
	1000	1.21	3.90	TL	0.39	1.54	TL	0.14	0.29	0.44	133601	269804	411130
	2000	64.65	12.77	209.30	10.76	11.33	6.06	0.60	1.21	1.81	272438	549814	836905
	5000	TL	478.32	ML	11.80	29.52	17.08	3.83	8.61	12.23	689237	1390675	2116720
6	100	0.07	0.18	0.32	0.04	0.08	0.17	0.00	0.00	0.00	12067	24538	36953
	500	2.28	2.51	19.48	0.29	0.66	3.41	0.03	0.07	0.11	61295	122941	184592
	1000	27.39	116.15	164.23	3.38	4.16	22.42	0.14	0.29	0.44	119912	240120	360361
	2000	24.64	826.12	627.04	6.58	2.61	10.32	0.59	1.18	1.78	242370	485031	727723
	5000	909.99	ML	ML	44.18	40.45	155.50	3.78	7.61	12.26	617052	1234392	1851776
7	100	0.07	0.15	0.24	0.04	0.05	0.10	0.00	0.00	0.00	12080	24564	36994
	500	1.25	0.85	9.14	0.56	2.13	0.23	0.03	0.07	0.11	61358	123068	184782
	1000	49.13	6.35	97.29	0.98	1.42	3.64	0.14	0.29	0.44	120034	240368	360732
	2000	897.75	TL	250.01	14.79	2.82	6.24	0.59	1.18	1.80	242616	485522	728458
	5000	TL	ML	ML	44.06	33.20	59.05	3.81	7.56	11.57	617678	1235644	1853652
8	100	0.35	0.01	0.02	0.02	0.00	0.01	0.00	0.00	0.00	14004	28210	42916
	500	0.59	0.29	1.04	0.18	0.19	0.34	0.03	0.07	0.11	68672	138644	210916
	1000	1.08	1.58	1.69	0.45	0.55	0.45	0.14	0.29	0.44	133722	270046	411168
	2000	2.15	4.57	5.87	0.57	1.53	1.91	0.59	1.18	1.79	269304	543910	828214
	5000	15.71	47.92	ML	3.18	25.67	23.57	3.79	7.71	12.59	684372	1381344	2102816
9	100	0.00	0.00	0.02	0.00	0.00	0.01	0.29	0.65	0.98	2921	11548	26568
	500	0.03	0.09	0.17	0.02	0.03	0.05	8.12	16.33	25.30	14563	58423	132701
	1000	0.17	0.44	0.66	0.07	0.12	0.18	33.46	66.06	98.96	29462	116287	265579
	2000	0.82	TL	TL	0.24	695.56	TL	130.85	264.36	396.21	61180	240024	547607
	5000	717.22	17.61	ML	23.30	2.73	4.26	826.20	TL	TL	159035	621589	1397221

Table 7: Runtime in seconds and best solution value for each instance with $\overline{R} = 10000$.

			$(\mathrm{F}_l)^H$			$(\mathrm{F}_e)^H$			DP			z^*	
cat.	$n \diagdown C$	25%	50%	75%	25%	50%	75%	25%	50%	75%	25%	50%	75%
1	100	3.19	20.29	4.68	1.35	20.22	2.13	0.02	0.04	0.06	21811	80166	179311
	500	53.62	33.51	209.00	4.86	12.02	57.95	0.46	0.89	1.32	114523	459156	1039992
	1000	43.44	834.24	49.45	21.51	181.38	10.50	1.71	3.38	5.16	224150	941497	2117198
	2000	330.56	807.16	423.48	60.51	178.36	64.81	6.95	13.34	21.25	483979	1925732	4295218
	5000	628.31	ML	ML	120.47	64.96	176.26	44.68	86.53	127.29	1179665	4686830	10507765
2	100	0.10	0.13	0.32	0.08	0.07	0.24	0.02	0.05	0.07	85229	185013	292897
	500	3.10	8.25	2.47	0.97	4.56	5.31	0.49	0.94	1.37	498746	1073343	1698472
	1000	10.71	57.56	33.52	3.55	7.58	19.88	1.81	3.72	5.56	1028577	2201879	3468177
	2000	145.06	144.01	685.58	18.73	36.83	90.64	7.15	14.44	21.59	2071531	4430501	6976442
	5000	492.25	ML	ML	24.54	134.98	112.60	44.24	89.06	130.24	5263091	11251628	17711417
3	100	0.17	53.55	296.85	0.33	80.08	34.27	0.01	0.02	0.04	118252	238588	363929
	500	2.80	36.61	2.67	0.73	237.60	4.20	0.36	0.71	1.13	649309	1310618	2000927
	1000	67.69	294.28	6.28	116.11	8.90	4.05	1.49	3.20	4.59	1351149	2726298	4156447
	2000	92.90	13.36	10.70	975.22	TL	3.46	6.13	13.42	19.62	2747609	5545219	8442829
	5000	ML	ML	ML	TL	TL	TL	40.73	82.14	126.33	6926796	13981593	21278389
4	100	0.12	6.73	20.12	0.08	5.43	12.39	0.03	0.05	0.08	79817	186007	297960
	500	TL	TL	TL	TL	TL	TL	0.51	1.04	1.57	476256	1068850	1693224
	1000	TL	TL	TL	TL	TL	TL	1.96	4.10	6.36	1010792	2247788	3541015
	2000	TL	TL	TL	TL	89.59	TL	7.83	16.77	24.67	2066109	4591558	7222366
	5000	ML	ML	ML	TL	TL	TL	51.66	104.00	155.23	5230252	11605856	18236923
5	100	0.17	5.31	1.80	138.17	2.47	1.79	0.01	0.03	0.05	114770	230670	352572
	500	TL	TL	TL	TL	TL	9.43	0.38	0.77	1.18	664526	1343087	2046816
	1000	47.03	TL	7.19	40.64	TL	4.01	1.53	3.05	4.75	1349293	2725604	4150205
	2000	343.94	220.77	247.50	95.10	28.10	7.73	6.53	13.19	19.48	2733192	5519666	8401427
	5000	TL	ML	ML	24.60	TL	17.32	40.33	82.60	124.00	6909253	13944843	21218933
6	100	0.08	0.26	0.04	0.06	0.14	0.05	0.01	0.02	0.04	101130	204809	308231
	500	4.78	32.60	36.67	1.13	3.75	12.74	0.36	0.72	1.08	580840	1163941	1747413
	1000	37.97	125.58	169.78	4.18	7.62	48.17	1.49	3.03	4.49	1213512	2429569	3645954
	2000	302.56	439.40	539.30	15.96	14.52	14.71	6.06	12.65	19.84	2474759	4952346	7430298
	5000	ML	ML	ML	909.91	404.44	TL	40.36	80.72	121.34	6248810	12500605	18752821
7	100	0.08	0.27	0.05	0.06	0.12	0.05	0.01	0.02	0.04	101156	204848	308280
	500	4.31	10.81	28.58	0.74	3.10	10.70	0.35	0.71	1.07	580892	1164068	1747602
	1000	42.64	200.22	115.74	3.85	13.58	35.39	1.49	3.16	4.88	1213636	2429818	3646326
	2000	168.13	279.62	845.65	13.94	90.86	43.43	6.64	12.56	19.33	2475004	4952838	7431034
	5000	ML	ML	ML	14.07	223.02	188.44	41.21	83.21	124.01	6249436	12501856	18754698
8	100	0.16	0.53	289.00	0.62	0.24	10.76	0.01	0.02	0.04	118262	237710	363974
	500	0.95	16.79	82.33	0.29	8.52	4.98	0.35	0.71	1.07	649372	1310744	2001116
	1000	39.14	280.84	226.95	1.52	5.70	24.75	1.49	2.98	4.59	1351272	2726546	4156818
	2000	5.90	288.46	28.49	5.09	TL	645.67	6.13	13.07	20.14	2747854	5545710	8443564
	5000	ML	25.03	ML	TL	5.87	7.71	40.53	87.43	120.95	6927422	13982844	21280266
9	100	0.00	0.00	0.01	0.00	0.00	0.01	3.31	6.73	10.06	18762	84731	207429
	500	0.05	0.11	1.12	0.02	0.04	0.67	83.89	176.32	247.69	145911	596495	1326550
	1000	8.79	4.10	0.84	0.56	2.03	0.23	346.45	TL	TL	302563	1216230	2428114
	2000	350.56	2.31	3.78	71.66	0.57	0.87	TL	TL	TL	628397	2428114	2428114
	5000	306.25	14.59	ML	36.89	3.18	6.06	TL	TL	TL	1577597	2428114	2428114

from an exact point of view. With this article we hope to stimulate the research on this new and challenging family of problems which can give rise to new solution properties and exact methods. Finally, it could be interesting to study other variants of the classical knapsack problem in this "lazy" setting, such as quadratic knapsack and covering problems [10, 19] or the multiple-choice knapsack problem [18].

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